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SOME ESTIMATES FOR MAXIMAL FUNCTIONS ON KÖTHE FUNCTION SPACES*

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ABSTRACT

Hardy–Littlewood maximal operators defined for X-valued functions on $\mathbb{R}^n$, X a Köthe function space, are considered. Necessary and sufficient conditions on weights $v(x)$ for the existence of a weight $u(x)$ such that the maximal operator is bounded from $L^p_x(v)$ to $L^p_x(u)$ are given. The cases of $X = l^r$ and $X = l^{s,r}$ are studied in detail. Also, the behavior of the weights under convexity assumptions on X is considered.

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Introduction

In 1978 B. Muckenhoupt, [M], posed the problem of characterizing the pair of weights $(u(x), v(x))$ for which the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\, dy$$

satisfies

$$\int_{\mathbb{R}^n} |Mf(x)|^p\, u(x)\, dx \le c \int_{\mathbb{R}^n} |f(x)|^p\, v(x)\, dx, \tag{0.1}$$

where $1 < p < \infty$.

As a preliminary step to the characterization of pairs of weight for (0.1), he proposed the characterization of the weights $u(x)$ (respectively $v(x)$) for which there exists a non-trivial weight $v(x)$ (respectively $u(x)$) such that (0.1) holds. Answers to the latter questions were given by L. Carleson and P. W. Jones [C,J], A. E. Gatto and C. Gutiérrez [G,G], W. S. Young [Y] and J. L. Rubio de Francia [RdeF]. These authors found that the condition

$$\int_{\mathbb{R}^n} u(x)(1+|x|)^{-np}\, dx < \infty \tag{0.2}$$

on the weight $u(x)$ is necessary and sufficient for the existence of a weight $v(x)$ satisfying (0.1). For the existence of $u(x)$ it is necessary and sufficient that $v(x)$ satisfy

$$\sup_{R\ge 1} R^{-np'} \int_{|x|\le R} v(x)^{-p'/p}\, dx < \infty. \tag{0.3}$$

The solution for the analogous problem for the Riesz transforms, that is to say, substituting $R_i f$ for Mf, is also known, [RdeF] and [C,J]. In this case (0.2) is again a necessary and sufficient condition on $u(x)$, while the necessary and sufficient condition on $v(x)$ is

$$\int_{\mathbb{R}^n} (1+|x|)^{-np'} v(x)^{-p'/p}\, dx < \infty. \tag{0.4}$$

We shall denote by D_p^* the class of all weights $v(x)$ satisfying (0.3) and by D_p the class of all weights satisfying (0.4). It is easy to see that $D_p \subsetneq D_p^*$.

The question of giving a characterization for the pair of weights (u, v) for which (0.1) holds was answered by E. Sawyer, [S]; the analogous problem for the Riesz transforms remains open. Also open is the problem of the characterization of the pairs of weights for which

$$\int_{\mathbb{R}^n}\left(\sum_{i=1}^{\infty} Mf_i(x)^r\right)^{p/r} u(x)\,dx \le c\int_{\mathbb{R}^n}\left(\sum_{i=1}^{\infty}|f_i(x)|^r\right)^{p/r} v(x)\,dx \tag{0.5}$$

holds. Recently, L. M. Fernández-Cabrera and J. L. Torrea [F,T] have shown that (0.2) is a necessary and sufficient conditions on $u(x)$ for the existence of a weight $v(x)$ such that (0.5) holds.

In this paper we give a necessary and sufficient condition on $v(x)$ in order that (0.5) hold for some non-trivial weight $u(x)$. We study the problem in the more general context of Hardy–Littlewood operators $\mathcal{M}f$ for X-valued functions $f(x)$, X a Banach lattice, see §1. Thus, we are led to consider inequalities of the type

$$\int_{\mathbb{R}^n}\|\mathcal{M}f(x)\|_X^p\, u(x)\,dx \le c\int_{\mathbb{R}^n}\|f(x)\|_X^p\, v(x)\,dx. \tag{0.6}$$

Clearly, when $X = l^r$ we obtain (0.5).

The main result of this paper is Theorem 1.7. It gives a characterization of the classes $D(p,X)$ of weights $v(x)$ for which there exists $u(x)$ such that (0.6) holds, under certain restrictions on the lattice X. It turns out that these classes of weights are intermediate classes between the classes D_p and D_p^* associated with the analogous problem for the Riesz and Hardy–Littlewood operators, respectively. The classes $D(p,X)$ can be defined, under certain restriction, for a Banach lattice X without using the Hardy–Littlewood operator $\mathcal{M}$ and we shall proceed in this way (see 1.4).

In §2, for the case $X = l^r$, we find a much more explicit characterization of the classes $D(p,l^r)$ that we denote simply as $D(p,r)$. Moreover, we prove that the inclusions

$$D_p = D(p,1) \subsetneq D(p,r) \subsetneq D(p,s) \subsetneq D(p,p) = D(p,t) = D_p^*$$

hold for $1 < r < s < p < t < \infty$, see Corollary 2.7.

In §3 we give some relations between $D(p,X)$ and $D(p,r)$ classes for spaces X with convexity properties.

1. Main results

Let $(\Omega, \Sigma, d\omega)$ be a complete σ-finite measure space. A Banach space X consisting of equivalence classes modulo equality almost everywhere of locally integrable real functions on Ω is called a Köthe function space if the following conditions hold:

(i) If $|f(\omega)| \leq |g(\omega)|$ a.e. on Ω, f is measurable and $g \in X$, then f belongs to X and $\|f\| \leq \|g\|$.

(ii) For every $E \in \Sigma$ with $\mu(E) < \infty$, the characteristic function $\chi_E(\omega)$ of E belongs to X.

Every Köthe function space is a Banach lattice under the natural order:

$$f \geq 0 \quad \text{if and only if} \quad f(\omega) \geq 0 \text{ a.e. on } \Omega.$$

Given a measurable function $a(\omega)$ on Ω such that, for every $f \in X$, $f(\omega)a(\omega)$ belongs to $L^1(d\omega)$, we define

$$x_a^*(f) = \int_\Omega f(\omega)a(\omega)\,d\omega.$$

The linear functional x_a^* turns out to be bounded on X and x_a^* is denoted by a. Any functional on X of the form x_a^* is called an integral and the linear space of all integrals is denoted by X'. The linear space X' is said to be norming if for every $f \in X$ we have

$$\|f\|_X = \sup_{\substack{\|a\|_{X^*} \leq 1 \\ a \in X'}} \int_\Omega f(\omega)a(\omega)\,d\omega.$$

For more information on Banach lattices and Banach function spaces we refer the reader to [L,T].

Let X be a Banach lattice and let J be a finite subset of the set $\mathbb{Q}_+$ of the positive rational numbers. Given a locally integrable X-valued function $f(x)$, x in $\mathbb{R}^n$, we define

$$\mathcal{M}_J f(x) = \sup_{r \in J} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|\,dy,$$

where $B(x,r)$ is the ball of radius r centered at x ($|f(y)| = f(y) \vee (-f(y))$ in the lattice X).

We shall assume in the sequel that X is a Köthe function space and X' norming. Then, for any $x \in \mathbb{R}^n$, $\mathcal{M}_J f(x)$ is a function of $\omega \in \Omega$ given by

$$\mathcal{M}_J f(x,\omega) = \sup_{r\in J} |B(x,r)|^{-1} \int_{B(x,r)} |f(y,\omega)|\, dy.$$

Here $|f(y,\omega)|$ is the absolute value for real numbers and the sup is the supremum for the lattice $\mathbb{R}$ of the real numbers.

We shall say that a Banach lattice satisfies the **Hardy–Littlewood property** (HL) if there exists p_0, $1 < p_0 < \infty$, such that the operators $\mathcal{M}_J$ are uniformly bounded on $L^{p_0}_X(\mathbb{R}^n)$, that is to say, the inequality

$$\int_{\mathbb{R}^n} \|\mathcal{M}_J f(x)\|_X^{p_0}\, dx \le c_{p_0} \int_{\mathbb{R}^n} \|f(x)\|_X^{p_0}\, dx$$

holds with a finite constant c_{p_0} not depending on J.

Given a Banach lattice X, a result of Bourgain [B] (see also [RdeF1]) says that $\mathcal{M}_J$ are bounded (uniformly on J) in $L^p_X(\mathbb{R}^n)$ and also in $L^{p'}_{X^*}(\mathbb{R}^n)$ for some p, $1 < p < \infty$, where p' is the exponent conjugate to p, if and only if X is U.M.D.

The (HL) property was introduced by J. García-Cuerva, R.A. Macías and J.L. Torrea in [G-C,M,T]. In this paper they proved that a Banach lattice X has the (HL) property if and only if

$$(1.1)\qquad \Big|\, \{x\colon \big(\sum_{k=1}^{\infty} \|\mathcal{M}_J f_k(x)\|_X^p\big)^{1/p} > \lambda\}\, \Big| \le c_p \lambda^{-1} \int \big(\sum_{k=1}^{\infty} \|f_k(x)\|_X^p\big)^{1/p}$$

holds for any p, $1 < p < \infty$, with c_p a finite constant not depending on J (see Theorem 1.7).

Definition 1.2: Given a finite sequence $\{r_i\}_{i=1}^m$, where $r_i \ge 1$ and $r_i \in \mathbb{Q}_+$, let us denote by B_i the balls $B(0,r_i)$. If $\{\Omega_i\}_{i=1}^m$ is a measurable partition of Ω, we define the function $\varphi(x,\omega)$ as

$$(1.3)\qquad \varphi(x,\omega) = \sum_{i=1}^{m} |B_i|^{-1} \chi_{B_i}(x) \chi_{\Omega_i}(\omega).$$

We observe that $\varphi(x,\omega)$ is a step function in ω, for any given x. Moreover, if a belongs to X' then for any given x, $\varphi(x,\omega)a(\omega)$ belongs to X'. We denote this function by $\varphi(x)a$.

Definition 1.4: Let $v(x)$ be a weight on $\mathbb{R}^n$ and X be a Köthe function space with X' norming. For $1 < p < \infty$, we shall say that $v(x)$ belongs to the class $D(p, X)$ if

$$\sup_{\substack{a \in X' \\ \|a\|_{X^*} \leq 1}} \int_{\mathbb{R}^n} \|\varphi(x)a\|_{X^*}^{p'} v(x)^{-p'/p} \, dx < \infty.$$

Next, we shall show that the classes D_p and D_p^* are in some sense extremal classes of weights. More precisely, the following proposition holds:

PROPOSITION 1.5: *Let X be a Köthe function space with X' norming. Then for any p, $1 < p < \infty$, we have*

$$D_p \subset D(p, X) \subset D_p^*.$$

Proof: Let $B = B(0, r)$, $r \geq 1$. Then, $\varphi(x, \omega)$ defined by $\varphi(x, \omega) = \chi_B(x)/|B|$ satisfies

$$\varphi(x, \omega) = \frac{\chi_B(x)}{c_n r^n} \leq \frac{2^n}{c_n} \frac{1}{(1 + |x|)^n}.$$

Therefore, $\varphi(x, \omega)$ is a function as in (1.3) and if $a \in X'$, $\|a\|_{X^*} \leq 1$, then

$$|\varphi(x)a| \leq c\, |a|(1 + |x|)^{-n}.$$

Since $\|a\|_{X^*} = \| \, (|a|) \, \|_{X^*}$, we get

$$\int_{\mathbb{R}^n} \|\varphi(x)a\|_{X^*}^{p'} v(x)^{-p'/p} \, dx \leq c \, \|a\|_{X^*}^{p'} \int \frac{v(x)^{-p'/p}}{(1 + |x|)^{np'}} \, dx,$$

which shows that $D_p \subset D(p, X)$.

To prove that $D(p, X) \subset D_p^*$, let $B = B(0, r)$, $r \geq 1$, and $\varphi(x, \omega) = \chi_B(x)/|B|$. Let $a \in X'$, $\|a\|_{X^*} = 1$, then

$$\|\varphi(x)a\|_{X^*} = \chi_B(x)/|B|.$$

Thus,

$$r^{-np'} \int_{|x| \leq r} v(x)^{-p'/p} \, dx = c_n \int_{\mathbb{R}^n} \|\varphi(x)a\|_{X^*}^{p'} v(x)^{-p'/p} \, dx,$$

which shows that $D(p, X) \subset D_p^*$. ∎

We shall see in §2 that D_p and D_p^* coincide with $D(p, l^1)$ and $D(p, l^\infty)$, respectively.

The following result gives a useful equivalence of the $D(p, X)$ condition on weights in terms of the maximal functions $\mathcal{M}_J$.

THEOREM 1.6: *Let X be a Köthe function space with X' norming and $1 < p < \infty$. Then, the following conditions are equivalent for a weight $v(x)$:*

(i) *$v(x)$ belongs to $D(p, X)$,*

(ii) *$v(x)^{-p'/p}$ is a locally integrable function on $\mathbb{R}^n$, and for any given ball $B = B(0, R)$, $R \geq 1$ and any X-valued locally integrable function $f(x)$ supported in the complement of $B(0, 2R)$, the inequality*

$$\sup_{x \in B} \|\mathcal{M}_J f(x)\|_X \leq c_R \left(\int_{\mathbb{R}^n} \|f(x)\|_X^p v(x)\, dx \right)^{1/p}$$

holds, with a finite constant c_R not depending on J.

Proof: Part (i) $\Longrightarrow$ (ii). Let $R \geq 1$, $R \in \mathbb{Q}_+$ and $\Omega_R = \Omega$. Then,

$$\varphi(x) = \varphi(x, \omega) = |B(0, R)|^{-1} \chi_{B(0,R)}(x)$$

is a function of the type of Definition 1.2.

Thus, since $\|a\varphi(x)\|_{X^*} = \|a\|_{X^*} |B(0, R)|^{-1} \chi_{B(0,R)}(x)$, we have

$$|B(0, R)|^{-p'} \int_{B(0,R)} v(x)^{-p'/p}\, dx \leq \sup_{\substack{\|a\|_{X^*} \leq 1 \\ a \in X'}} \int \|a\varphi(x)\|_{X^*}^{p'} v(x)^{-p'/p}\, dx \leq c,$$

which shows that $v(x)^{-p'/p}$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let $R \geq 1$ and let $f(x)$ be an X-valued function with support contained in $\mathbb{R}^n \setminus B(0, 2R)$ and belonging to $L^p_X(v)$. Let J be a finite set of positive rational numbers. We observe that if every $r \in J$ is smaller than or equal to R, then $\mathcal{M}_J f(x) = 0$ for x in $B(0, R)$ and, therefore, there is nothing to prove in this case. Let us assume that there exists at least one $r \in J$ such that $r > R$. We define $J' = \{s \colon s = 2r,\ r \in J,\ r > R\}$. If $x \in B(0, R)$, we have

$$\mathcal{M}_J f(x, \omega) = \sup_{\substack{r \in J \\ r > R}} \frac{1}{c_n r^n} \int_{B(x,r)} |f(y, \omega)|\, dy,$$

and, since for $r \in J$ and $r > R$ we get that $x \in B(0, R)$ implies $B(x, r) \subset B(0, 2r)$, then

$$\mathcal{M}_J f(x, \omega) \leq 2^n \sup_{s \in J'} \frac{1}{c_n s^n} \int_{B(0,s)} |f(y, \omega)| dy.$$

Now, let Ω_s, $s \in J'$, be the subset of Ω where the supremum on the second member is attained for that s. Then, if

$$\varphi(x, \omega) = \sum_{s \in J'} |B(0, s)|^{-1} \chi_{B(0,s)}(x) \chi_{\Omega_s}(\omega),$$

we have

$$\mathcal{M}_J f(x,\omega) \le c \int_{\mathbb{R}^n} |f(y,\omega)|\varphi(y,\omega)\,dy.$$

Thus, for $x \in B(0,R)$, we get

$$\begin{aligned}\|\mathcal{M}_J f(x)\|_X &\le c \sup_{\substack{a\in X' \\ \|a\|_{X^*}\le 1}} \int_\Omega a(\omega) \int_{\mathbb{R}^n} f(y,\omega)\varphi(y,\omega)\,dy\,d\omega \\ &\le c \sup_{\substack{a\in X' \\ \|a\|_{X^*}\le 1}} \int_{\mathbb{R}^n} \|f(y)\|_X \cdot \|a\varphi(y)\|_{X^*}\,dy.\end{aligned}$$

By Hölder's inequality and (i), we obtain

$$\begin{aligned}\|\mathcal{M}_J f(x)\|_X &\le \left(\int \|f(y)\|_X^p v(y)\,dy\right)^{1/p} \left(\int \|a\varphi(y)\|_{X^*}^{p'} v(y)^{-p'/p}\,dy\right)^{1/p'} \\ &\le c\left(\int \|f(y)\|_X^p v(y)\,dy\right)^{1/p}.\end{aligned}$$

This ends the proof of (i) $\Longrightarrow$ (ii).

Let us consider now part (ii) implies (i). By Hölder's inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} \|a\varphi(x)\|_{X^*}^{p'} v(x)^{-p'/p}\,dx \\ &\qquad= \int_{\mathbb{R}^n} \|a\varphi(x)v(x)^{-1/p}\|_{X^*}^{p'}\,dx \\ &\qquad= \sup_{\|g\|_{L^p_X}\le 1} \left(\int_{\mathbb{R}^n} \left(\int_\Omega a(\omega)\varphi(x,\omega)g(x,\omega)\,d\omega\right) v(x)^{-1/p}\,dx\right)^{p'} \\ &\qquad\le (\sup I_1 + \sup I_2)^{p'},\end{aligned}$$

where

$$\begin{aligned}I_1 &= \int_{B(0,2R)} \left(\int_\Omega a(\omega)\varphi(x,\omega)g(x,\omega)\,d\omega\right) v(x)^{-1/p}\,dx \\ &\le \int_{B(0,2R)} \|g(x)\|_X v(x)^{-1/p}\,dx \\ &\le \left(\int \|g(x)\|^p\,dx\right)^{1/p} \cdot \left(\int_{B(0,2R)} v(x)^{-p'/p}\,dx\right)^{1/p'} \\ &\le c_R < \infty.\end{aligned}$$

Let J be the finite set of rational numbers involved in the definition of $\varphi(x,\omega)$. Then

$$I_2 = \int_{\mathbb{R}^n}\left(\int_\Omega a(\omega)\varphi(x,\omega)g(x,\omega)\chi_{\mathbb{R}^n\setminus B(0,2R)}(x)\,d\omega\right)v(x)^{-1/p}\,dx$$
$$\leq \int_\Omega |a(\omega)|\mathcal{M}_J\left(g(x,\omega)\chi_{\mathbb{R}^n\setminus B(0,2R)}(x)v(x)^{-1/p}\right)(0,\omega)\,d\omega.$$

By condition (ii), we get

$$I_2 \leq \left\|\mathcal{M}_J\big(g(x,\omega)\chi_{\mathbb{R}^n\setminus B(0,2R)}(x)v(x)^{-1/p}\big)(0)\right\|_X$$
$$\leq c\left(\int \|g(x)v(x)^{-1/p}\|_X^p v(x)\,dx\right)^{1/p} = c\left(\int \|g(x)\|_X^p\,dx\right)^{1/p},$$

which ends the proof of part (ii) implies (i). ∎

THEOREM 1.7: *Let X be a Köthe function space having the (HL) property, with X' norming. Then for every p, $1 < p < \infty$, the following conditions are equivalent:*

(i) *$v(x)$ belongs to the class $D(p,X)$.*

(ii) *There exist a non-trivial weight $u(x)$ and a finite constant c_p, not depending on J, such that*

$$\int_{\mathbb{R}^n}\|\mathcal{M}_J f(x)\|_X^p u(x)\,dx \leq c_p\int_{\mathbb{R}^n}\|f(x)\|_X^p v(x)\,dx$$

holds for every f in $L^p_X(v)$.

Proof: First we shall prove (ii) $\Longrightarrow$ (i). Let $R>0$, such that $\int_{B(0,R)} u(x)\,dx > 0$. We can always assume that the function $u(x)$ is bounded on $\mathbb{R}^n$ and therefore integrable on $B(0,R)$. Let a belong to X', $\|a\|_{X^*}\leq 1$, and $\varphi(x,\omega)$ as in (1.3). We have

$$\int_{\mathbb{R}^n}\|\varphi(y)a\|_{X^*}^{p'}\,v(y)^{-p'/p}\,dy$$
$$= \left[\sup_{\|g\|_{L^p_X}\leq 1}\int_\Omega a(\omega)\left(\int_{\mathbb{R}^n}\varphi(y,\omega)v(y)^{-p'/p}g(y,\omega)\,dy\right)d\omega\right]^{p'}$$
$$\leq \sup_{\|g\|_{L^p_X}\leq 1}\Big\|\int_{\mathbb{R}^n}\varphi(y,\omega)v(y)^{-1/p}g(y,\omega)\,dy\,\Big\|_X^{p'}.$$

We shall estimate the integral in the last member of the inequality above. Let $r_1, \ldots, r_m$ be rational numbers greater than or equal to one and $\{\Omega_i\}_{i=1}^m$ the partition of Ω in the definition of $\varphi(x,\omega)$ given in (1.3). Then

$$\left|\int_{\mathbb{R}^n} \varphi(y,\omega)g(y,\omega)v(y)^{-1/p}\,dy\right| \tag{1.8}$$
$$\leq \sum_{i=1}^m \chi_{\Omega_i}(\omega)|B(0,r_i)|^{-1}\int_{B(0,ri)} |g(y,\omega)|v(y)^{-1/p}\,dy.$$

Let $z \in B(0,R)$. If $y \in B(0,r_i)$ it follows that $|z-y| \leq |z| + r_i \leq (R+1)r_i$. Then, (1.8) is less than or equal to

$$(R+1)^n \mathcal{M}_J(gv^{-1/p})(z),$$

where $J = \{(R+1)r_1, \ldots, (R+1)r_m\}$. Thus, using the hypothesis (ii) we get that the X-norm of (1.8) is bounded by

$$(R+1)^n\left[u(B(0,R))^{-1}\int \|\mathcal{M}_J(gv^{-1/p})(z)\|_X^p u(z)\,dz\right]^{1/p}$$
$$\leq c\left(\int \|g(y)\|_X^p v(y)^{-1}v(y)\,dy\right)^{1/p} = c\,\|g\|_{L_X^p} \leq c.$$

This proves that (ii) implies (i).

Now, let us consider (i) $\Longrightarrow$ (ii). Let $B_k = B(0,2^k)$; $S_k = B_k - B_{k-1}$, $k = 1,2,\ldots$ and $S_0 = B_0$. As we shall see, it is enough to prove that for any sequence $\{J_j\}$ and s, $0 < s < 1 < p$, the inequality

$$\left\|\left(\sum_j \|\mathcal{M}_{J_j} f_j(x)\|_X^p\right)^{1/p}\right\|_{L^s(S_k)} \leq c_k \left(\sum_j \|f_j\|_{L_X^p(v)}^p\right)^{1/p} \tag{1.9}$$

holds with finite constants c_k independent of the sets J_j. Les us proceed to prove (1.9). Given $k \geq 0$, let

$$f_j'(x) = f_j(x)\chi_{B_{k+1}}(x) \quad \text{and} \quad f_j'' = f_j - f_j'.$$

Since the space X has the (HL)-property, by (1.1), we have

$$|\{x\colon \left(\sum_j \|\mathcal{M}_J f_j'(x)\|_X^p\right)^{1/p} > \lambda\}| \leq c_p\lambda^{-1}\int_{B_{k+1}}\left(\sum_j \|f_j(x)\|_X^p\right)^{1/p} dx,$$

where the constant c_p does not depend on J. By a suitable version of the Kolmogorov inequality, we get

$$\| \left(\sum_j \|\mathcal{M}_J f_j'(x)\|_X^p \right)^{1/p} \|_{L^s(B_k)} \leq c|B_k|^{1/s-1} \int_{B_{k+1}} \left(\sum_j \|f_j(x)\|_X^p \right)^{1/p} dx.$$

From the hypotheses (i) it follows that the right hand side of this inequality is bounded by

$$c\,|B_k|^{1/s-1} \left(\int_{B_{k+1}} (\sum_j \|f_j(x)\|_X^p) v(x)\,dx \right)^{1/p} \cdot \left(\int_{B_{k+1}} v^{-p'/p}(x)\,dx \right)^{1/p'}$$
$$\leq c_k \left(\sum_j \|f_j\|_{L_X^p(v)}^p \right)^{1/p}.$$

The constants c_k are finite since, as we have shown in Theorem 1.6, under thecondition (i) the function $v(x)^{-p'/p}$ is locally integrable on $\mathbb{R}^n$.

In order to prove (1.9) for the sequence of functions $\{f_j''\}$ we observe that by Theorem 1.6, part (ii), we have

$$\sup_{x\in B_k} \|\mathcal{M}_J f_j''(x)\|_X \leq d_k \left(\int \|f_j''(z)\|_X^p v(z)\,dz \right)^{1/p}$$
$$\leq d_k \left(\int \|f_j(z)\|_X^p v(z)\,dz \right)^{1/p}.$$

Thus,

$$\left(\int_{B_k} (\sum_j \|\mathcal{M}_J f_j''(x)\|_X^p)^{s/p}\,dx \right)^{1/s} \leq c_k \left(\sum_j \|f_j\|_{L_X^p(v)}^p \right)^{1/p},$$

where $c_k = d_k|B_k|^{1/s}$. Gathering together our estimates for the sequences $\{f_j'\}$ and $\{f_j''\}$, we get

$$\| \left(\sum_j \|\mathcal{M}_J f_j(x)\|_X^p \right)^{1/p} \|_{L^s(S_k)} \leq c_k \left(\sum \|f_j\|_{L_X^p(v)}^p \right)^{1/p}, \tag{1.10}$$

where the constant c_k does not depend on J.

In order to obtain (1.9) from (1.10) we just observe that for any finite sum

$$\sum_{j=1}^{m} \|\mathcal{M}_{J_j} f_j(x)\|_X^p \leq \sum_{j=1}^{m} \|\mathcal{M}_J f_j(x)\|_X^p,$$

where $J = \bigcup_{j=1}^{m} J_j$.

Finally, the sufficiency of (1.9) to prove (ii) follows from a version of a theorem of Rubio de Francia that can be found in [F,T]. For the sake of completeness, we give a direct argument to show that (1.9) implies (ii).

Let $T_j^k f(x) = \|\chi_{S_k}(x)\mathcal{M}_{J_j} f(x)\|_X$. The operators of the family $\{T_j^k\}$, k fixed, are sublinear and, by (1.9), they satisfy the hypotheses of Theorem 4.2 on p. 554 of the book [G-C,RdeF]. Therefore, there exists $u_k(x) > 0$ such that for every j,

$$\left(\int |T_j^k f(x)|^p u_k(x)\, dx\right)^{1/p} \leq c_k \left(\int \|f(x)\|_X^p v(x)\, dx\right)^{1/p}$$

holds for a finite constant c_k not depending on j. Thus, for $J = J_j$

$$\left(\int_{S_k} \|\mathcal{M}_J f(x)\|_X^p u_k(x)\, dx\right)^{1/p} \leq c_k \|f\|_{L_X^p(v)}.$$

Multiplying this inequality by $c_k^{-p} 2^{-k}$ and adding in k we get

$$\left(\int \|\mathcal{M}_J f(x)\|_X^p u(x)\, dx\right)^{1/p} \leq \|f\|_{L_X^p(v)},$$

where $u(x) = \sum_k c_k^{-p} 2^{-k} u_k(x) \chi_{S_k}(x) > 0$ a.e. ∎

2. Applications to l^r-valued functions

We denote by l^r, $0 < r \leq \infty$ the linear space of all the sequences $a = \{a_k\}_{k=1}^{\infty}$ of real numbers such that

$$\|a\|_r = \left(\sum_{k=1}^{\infty} |a_k|^r\right)^{1/r} < \infty,$$

whenever $0 < r < \infty$ and, for $r = \infty$,

$$\|a\|_\infty = \sup_k |a_k|.$$

If $1 \le r \le \infty$, then l^r equipped with the norm $\|a\|_r$ is a Banach space. It can be shown that for $1 < r \le \infty$, the spaces l^r have the (HL) property, see section 2 of [G-C,M,T].

Let $f(x) = \{f_i(x)\}$ be an l^r-valued and locally integrable function on $\mathbb{R}^n$. We define

$$\mathcal{M}f(x) = \{Mf_i(x)\},$$

where M stands for the ordinary Hardy–Littlewood maximal operator for scalar valued functions on $\mathbb{R}^n$. Let J be any finite subset of the set $\mathbb{Q}_+$ of positive rational numbers. Then

$$\mathcal{M}_J f(x) = \{M_J f_i(x)\},$$

where M_J is the operator for the lattice of the real numbers. It is easy to prove that

$$\|\mathcal{M}f(x)\|_r = \sup_J \|\mathcal{M}_J f(x)\|_r.$$

Moreover, if $\{J_k\}$ is a non-decreasing sequence of subsets of $\mathbb{Q}_+$ satisfying $\mathbb{Q}_+ = \bigcup J_k$, then

$$\|\mathcal{M}f(x)\|_r = \lim_{k\to\infty} \|\mathcal{M}_{J_k} f(x)\|_r. \tag{2.1}$$

We observe that the sequence in the right hand side is not decreasing. The class $D(p, l^r)$ shall be denoted by $D(p, r)$.

We shall need the following well known result.

LEMMA 2.2: *If $1 \le t \le \infty$, then*

$$\sup_{\|a\|_t \le 1} \Big|\sum_k A_k a_k\Big| = \|A\|_{t'}.$$

If $0 < t \le 1$, then

$$\sup_{\|a\|_t \le 1} \Big|\sum_k A_k a_k\Big| = \|A\|_{\infty}.$$

This lemma shows that l^t is norming when considered as a subspace of $(l^{t'})^*$, $1 \le t \le \infty$ or as a subspace of $(l^\infty)^*$, $0 < t \le 1$.

Since for $1 < r \le \infty$, the space l^r has the (HL) property, then Theorem 1.7, applied to $X = l^r$, $1 < r \le \infty$ shows that $v(x)$ belongs to $D(p, r)$, $1 < p < \infty$, if and only if there exists a non-trivial weight $u(x)$ such that

$$\int \|\mathcal{M}_J f(x)\|_r^p u(x)\,dx \le c_p \int \|f(x)\|_r^p v(x)\,dx,$$

where the constant c_p does not depend on J.

Thus, by (2.1) and the monotone convergence theorem, we get

$$\int \|\mathcal{M}f(x)\|_r^p u(x)\,dx \le c \int \|f(x)\|_r^p v(x)\,dx.$$

Hence, we have proved the following theorem:

THEOREM 2.3: *A weight $v(x)$ belongs to $D(p,r)$, $1 < p < \infty$, $1 < r \le \infty$ if and only if there exists a non-trivial weight $u(x)$ such that*

$$\int_{\mathbb{R}^n} \left(\sum_k Mf_k(x)^r \right)^{p/r} u(x)\,dx \le c \int_{\mathbb{R}^n} \left(\sum_k |f_k(x)|^r \right)^{p/r} v(x)\,dx.$$

Next, we shall study some properties of the classes $D(p,r)$. We begin with the following proposition rephrasing the $D(p,l^r) = D(p,r)$ condition:

PROPOSITION 2.4: *Let $B_k = B(0,2^k)$, $k = 0,1,2,\ldots$. A weight $v(x)$ belongs to $D(p,r)$, $1 \le r \le \infty$, $1 < p < \infty$, if and only if there exists a finite constant c such that*

$$\int_{\mathbb{R}^n} \left(\sum_{k=1}^{\infty} \left(b_k \chi_{B_k}(x) 2^{-nk} \right)^{r'} \right)^{p'/r'} v(x)^{-p'/p}\,dx \le c$$

holds for any sequence $b = \{b_k\}$, $b_k \ge 0$, $\|b\|_{r'} \le 1$.

Proof: Given a sequence $\{r_i\}$ of rational numbers greater than or equal to one, we define the sets

$$I(k) = \{i \colon 2^k \le r_i < 2^{k+1}\},$$

$k = 0,1,2,\ldots$. Then, if $a = \{a_i\} \in l^{r'}$, $1 \le r \le \infty$, $a_i \ge 0$, $\|a\|_{r'} \le 1$, we have

$$\sum_i (a_i \chi_{B(0,r_i)}(x) r_i^{-n})^{r'} \le \sum_k \Big(\sum_{i \in I(k)} a_i^{r'} \Big) \chi_{B_{k+1}}(x) 2^{-nkr'}.$$

Let us define $b = \{b_k\}$, where

$$b_k = \left(\sum_{i \in I(k)} a_i^{r'} \right)^{1/r'} ;$$

then $\|b\|_{r'} = \|a\|_{r'} \le 1$. This finishes the proof of the proposition. ∎

Now we give a characterization of the $D(p,r)$ condition in terms of averages of $v(x)^{-p'/p}$. This is the main result in this section.

Let $S_0 = B(0,1)$ and $S_k = B(0,2^k) \setminus B(0,2^{k-1})$, $k = 1,2,\ldots$.

THEOREM 2.5: *A weight $v(x)$ belongs to $D(p,r)$, $1<p<\infty$, $1\le r\le\infty$, if and only if the sequence $A=\{A_k\}$,*

$$A_k = 2^{-np'k}\int_{S_k} v(x)^{-p'/p}\,dx,$$

satisfies

(i) *$A\in l^{(r'/p')'}$ if $1\le r\le p$, and*
(ii) *$A\in l^{\infty}$ if $p\le r\le\infty$.*

Proof: For any sequence $a=\{a_i\}$, $a_i\ge 0$, we have

$$\begin{aligned}\left(\sum_k \left(a_k\chi_{B_k}(x)2^{-nk}\right)^{r'}\right)^{p'/r'} &= \left(\sum_m \chi_{S_m}(x)\sum_{k=m}^{\infty} a_k^{r'}2^{-nr'k}\right)^{p'/r'}\\ &= \sum_m \chi_{S_m}(x)2^{-nmp'}\left(2^{nr'm}\sum_{k=m}^{\infty} a_k^{r'}2^{-nr'k}\right)^{p'/r'}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^n}\left(\sum_k\left(a_k\chi_{B_k}(x)2^{-nk}\right)^{r'}\right)^{p'/r'} v(x)^{-p'/p}\,dx = \sum_m A_m b_m, \tag{2.6}$$

where

$$b_m = \left(2^{nr'm}\sum_{k=m}^{\infty} a_k^{r'}2^{-nr'k}\right)^{p'/r'}.$$

For the sequence $\{b_m\}$, we have

$$\begin{aligned}\sum_m b_m^{r'/p'} &= \sum_m 2^{nr'm}\sum_{k\ge m} a_k^{r'}2^{-nr'k} = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}2^{nr'm}\right)a_k^{r'}2^{-nr'k}\\ &\le c\sum_{k=0}^{\infty} a_k^{r'}.\end{aligned}$$

Thus, $\|b\|_{r'/p'}\le c\|a\|_{r'}^{p'}$. By Lemma 2.2 and (2.6) we get

(i) if $1\le r\le p$, $1<p<\infty$ and A belongs to $l^{(r'/p')'}$, then $v(x)$ belongs to $D(p,r)$, and
(ii) if $p\le r\le\infty$, $1<p<\infty$ and A belongs to l^{∞}, then $v(x)$ belongs to $D(p,r)$.

Now, let $c = \{c_k\}$ be a sequence of real numbers such that $c_k \geq 0$ and $\|c_k\|_{r'/p'} \leq 1$. We define

$$a_k = c_k^{1/p'}.$$

Then, $a = \{a_k\}$ satisfies $\|a\|_{r'} \leq 1$. Thus

$$b_m = \left(2^{nr'm} \sum_{k=m}^{\infty} a_k^{r'} 2^{-nr'k}\right)^{p'/r'} \geq a_m^{p'} = c_m.$$

If $v(x)$ belongs to $D(p,r)$, then we have

$$\sum c_m A_m \leq \sum b_m A_m \leq c.$$

By Lemma 2.2 it follows that if $1 \leq r \leq p$ then A belongs to $l^{(r'/p')'}$, and if $p \leq r \leq \infty$ then A belongs to l^∞. ∎

COROLLARY 2.7: *If $1 < p < \infty$ and $1 < r < s < p$, then*

$$D_p = D(p,1) \subsetneq D(p,r) \subsetneq D(p,s) \subsetneq D(p,p) = D_p^*.$$

If $p \leq r \leq \infty$, then

$$D(p,r) = D_p^*.$$

Proof: By Proposition 1.5, we already know that $D_p \subset D(p,1)$ and $D(p,r) \subset D_p^*$. Let $v(x)$ belong to $D(p,1)$. By Theorem 2.5 the sequence $\{A_k\}$ belongs to l^1. Then

$$\int_{\mathbb{R}^n} \frac{v(x)^{-p'/p}}{(1+|x|)^{np'}}\,dx \leq \sum_{k=0}^{\infty} 2^{-np'k} \int_{S_k} v(x)^{-p'/p}\,dx = \sum_{k=0}^{\infty} A_k \leq c,$$

showing that $D(p,1) \subset D_p$.

Let $v(x)$ belong to D_p^*. Since

$$A_k = 2^{-np'k} \int_{S_k} v(x)^{p'/p}\,dx \leq 2^{-np'k} \int_{B_k} v(x)^{-p'/p}\,dx,$$

we have $\{A_k\} \in l^\infty$ and by Theorem 2.5, it follows that $v(x)$ belongs to $D(p,r)$ for $\infty \geq r \geq p$. Thus $D_p^* \subset D(p,r)$. ∎

3. The classes $D(p, X)$ for spaces X with convexity properties

We begin with the following definition.

Definition 3.1: A Köthe function space is said to be p-convex, $1 \le p < \infty$, if there exists a finite constant c_p such that

$$\Big\| \Big(\sum_i |x_i|^p \Big)^{1/p} \Big\|_X \le c_p \left(\sum_i \|x_i\|_X^p \right)^{1/p}$$

holds for any finite sequence $\{x_i\}$ of elements of X. The least constant c_p is called the p-convexity constant of X.

A Köthe function space is said to be q-concave if there exists a finite constant C^q such that

$$\left(\sum_i \|x_i\|_X^q \right)^{1/q} \le C^q \Big\| \Big(\sum_i |x_i|^q \Big)^{1/q} \Big\|_X, \qquad \text{if} \quad 1 < q < \infty$$

or

$$\max_i \|x_i\|_X \le C^\infty \Big\| \sup_i |x_i| \Big\|_X, \quad \text{if} \quad q = \infty,$$

holds for any finite sequence $\{x_i\}$ of elements of X. The last constant C^q is called the q-concavity constant of X.

It is known that X is p-convex (concave) if and only if X^* is q-concave (convex) and $c_p(X) = C^q(X^*)$, $(C^q(X) = c_p(X^*))$, $\frac{1}{p} + \frac{1}{q} = 1$.

By combining Theorem 1.f.7 and 1.f.12 in [L,T] we obtain the following useful result:

PROPOSITION 3.2: *Let X be a Köthe function space which is not p-convex for any p, $1 < p < \infty$. Then, for any $\varepsilon > 0$ and every positive integer m there exists a sequence $\{e_i\}_{i=1}^m$ of pairwise disjoint ($|e_i| \wedge |e_j| = 0$ if $i \neq j$) elements of X such that*

$$(1 - \varepsilon) \sum_{i=1}^m a_i \le \Big\| \sum_{i=1}^m a_i e_i \Big\|_X \le \sum_{i=1}^m a_i$$

holds for any choice of the sequence $\{a_i\}_{i=1}^m$ of non-negative scalars.

We observe that since the elements of the sequence $\{e_i\}_{i=1}^m$ are disjoint, then $|\sum a_i e_i| = \sum |a_i|\,|e_i|$. Therefore, the elements e_i may be assumed to be positive.

Now we are in position to state our first result concerning the classes $D(p, X)$.

THEOREM 3.3: *Let X be a Köthe function space with X' norming. Then, we have*

(i) *$D(p,X) = D_p$ for every p, $1 < p < \infty$, if and only if X is not r-convex for any $1 < r < \infty$,*

(ii) *$D(p,X) = D_p^*$ for some p, $1 < p < \infty$, if and only if X is r-convex for some $r > 1$.*

Proof: We shall prove first the "if" parts of (i) and (ii). Let us assume that X is not r-convex for any $1 < r < \infty$. Since by Proposition 1.5 we already know that $D_p \subset D(p,X)$ for any Köthe function space X and by Corollary 2.7 that $D_p = D(p,1)$, we only need to show that $D(p,X) \subset D(p,1)$.

Let $F(x) = \{f_i(x)\}$ be a sequence of real valued functions such that $f_i(x) \equiv 0$ if $i > m$ and satisfying

$$\int \Big(\sum_{i=1}^{m} |f_i(x)|\Big)^p v(x)\,dx < \infty.$$

Then, by Proposition 3.2 there exists a sequence $\{e_i\}_{i=1}^{m}$ of positive and pairwise disjoint elements of X such that

$$(1-\varepsilon)\sum_{i=1}^{m} |f_i(y)| \le \Big\|\sum_{i=1}^{m} |f_i(y)|e_i\Big\|_X \le \sum_{i=1}^{m} |f_i(y)|. \tag{3.4}$$

Let $G(y) = \sum_{i=1}^{m} |f_i(y)|e_i$. We have

$$\begin{aligned}\mathcal{M}_J G(x,\omega) &= \sup_{r\in J} |B(x,r)|^{-1} \int_{B(x,r)} \left(\sum_{i=1}^{m} |f_i(y)|e_i(\omega)\right) dy \\ &= \sup_{r\in J} |B(x,r)|^{-1} \sum_{i=1}^{m} \left(\int_{B(x,r)} |f_i(y)|\,dy\right) e_i(\omega).\end{aligned}$$

Since the elements $\{e_i\}$ are pairwise disjoint, the supports of the $e_i(\omega)$ are pairwise disjoint sets and therefore

$$\mathcal{M}_J G(x,\omega) = \sum_{i=1}^{m} M_J f_i(x) e_i(\omega).$$

Thus, by Proposition 3.2, we have

$$\begin{aligned}\|\mathcal{M}_J F(x)\|_{l^1} &= \sum_{i=1}^{m} M_J f_i(x) \le (1-\varepsilon)^{-1} \Big\|\sum_{i=1}^{m} M_J f_i(x) e_i\Big\|_X \\ &= (1-\varepsilon)^{-1} \|\mathcal{M}_J G(x)\|_X.\end{aligned}$$

Let the support of $F(x)$, and hence the support of $G(x)$, be contained in $\mathbb{R}^n \setminus B(0,2R)$. Since $v(x)$ belongs to $D(p,X)$, by Theorem 1.6 and (3.4), we get

$$\begin{aligned}\sup_{x\in B(0,R)} \|\mathcal{M}_J F(x)\|_{l^1} &\le (1-\varepsilon)^{-1} \sup_{x\in B(0,R)} \|\mathcal{M}_J G(x)\|_X \\ &\le c(R)(1-\varepsilon)^{-1}\left(\int_{\mathbb{R}^n} \|G(x)\|_X^p v(x)\,dx\right)^{1/p} \\ &\le c(R)(1-\varepsilon)^{-1}\left(\int \|F(x)\|_{l^1}^p v(x)\,dx\right)^{1/p}. \qquad (3.5)\end{aligned}$$

On the other hand, since $v \in D(p,X)$, by Theorem 1.6, we know that $v(x)^{-p'/p}$ is a locally integrable function, therefore, by Theorem 1.6 again, we have that (3.5) implies that $v(x)$ belongs to $D(p,1)$.

Let us prove the "if" part of (ii). We observe that if f belongs to L^r_X and X is assumed to be r-convex for a given r, $1<r<\infty$, we have

$$\|(\int |f(x,\omega)|^r\,dx)^{1/r}\|_X \le c_r\left(\int \|f(x)\|_X^r\,dx\right)^{1/r}. \qquad (3.6)$$

Let $v(x)$ belong to $D_r^* = D(r,\infty)$. By Hölder's inequality, we obtain

$$\begin{aligned}&\left(\int_{\mathbb{R}^n} \|\varphi(x)a\|_{X^*}^{r'} v(x)^{-r'/r}\,dx\right)^{1/r'} \\ &= \sup_{\|f\|_{L^r_X}\le 1} \left|\int\int \varphi(x,\omega)a(\omega)v(x)^{-1/r} f(x,\omega)\,d\omega\,dx\right| \\ &\le \int \left(\int \varphi(x,\omega)^{r'}|a(\omega)|^{r'} v(x)^{-r'/r}\,dx\right)^{1/r'}\cdot\left(\int |f(x,\omega)|^r\,dx\right)^{1/r} d\omega \\ &\le \left\|\left(\int |\varphi(x)a|^{r'} v(x)^{-r'/r}\,dx\right)^{1/r'}\right\|_{X^*}\cdot\left\|\left(\int |f(x,\omega)|^r\,dx\right)^{1/r}\right\|_X.\end{aligned}$$

Thus, by (3.6), we obtain

(3.7)

$$\left(\int_{\mathbb{R}^n} \|\varphi(x)a\|_{X^*}^{r'} v(x)^{-r'/r}\,dx\right)^{1/r'} \le c_r\left\|\left(\int |\varphi(x)a|^{r'} v(x)^{-r'/r}\,dx\right)^{1/r'}\right\|_{X^*}.$$

Let $\{\Omega_i\}_{i=1}^m$ be the partition of Ω and $B(0,r_i)$, $\{r_i\}_{i=1}^m$, $r_i \ge 1$ in the definition of $\varphi(x,\omega)$. Then, the integral inside the norm on the right hand side of the inequality above is bounded in the order by

$$\int \big| \sum a(\omega)\chi_{\Omega_i}(\omega)\chi_{B(0,r_i)}(x)/|B(0,r_i)| \big|^{r'} v(x)^{-r'/r}\, dx$$
$$\leq \sum_{i=1}^{m} |a(\omega)| \left(|B(0,r_i)|^{-r'} \int_{B(0,r_i)} v(x)^{-r'/r}\, dx\right) \chi_{\Omega_i}(\omega)$$
$$\leq c|a(\omega)|,$$

since $v \in D(r,\infty) = D_r^*$. Thus, (3.7) is bounded by a constant. This shows that $v(x)$ belongs to $D(r,X)$ or $D_r^* \subset D(r,X)$.

To prove the "only if" part of (i), let us assume that $D(p,X) = D_p$; then X cannot be p-convex since, by the "if" part of (ii), $D(p,X)$ would be equal to D_p^* and we know that $D_p \neq D_p^*$. Finally, if $D(p,X) = D_p^*$ for some p, $1 < p < \infty$ then X has to be r-convex for some $r > 1$, since otherwise $D(p,X) = D_p$ and, again, this is impossible. ∎

PROPOSITION 3.8: *Let X, Y be Köthe function spaces with X', Y' norming and with the property that if $a_1, \ldots a_n$ are disjoint positive elements of X' with $\|\sum a_j\|_{X^*} = 1$, then there exist $b_1, \ldots, b_n \in Y'_+$ so that $\|\sum b_j\|_{Y^*} = 1$ and $\|\sum \lambda_j b_j\|_{Y^*} \geq \|\sum \lambda_j a_j\|_{X^*}$ for all (λ_j). Then $D(p,Y) \subset D(p,X)$.*

Proof: We observe that if $\|a\|_{X^*} = 1$ and $\phi = \sum_{i=1}^m |B_i|^{-1}\chi_{B_i}(x)\chi_{\Omega_i}(\omega)$ is a step function one can consider $a_j = a\chi_{\Omega_j}$ and pick $b_1, \ldots, b_n$ supported on Ω_j' according to the theorem. Then let $\phi' = \sum |B_i|^{-1}\chi_{B_i}(x)\chi_{\Omega_j'}(\omega)$. Note that $\|\phi(x)a\|_{X^*} \leq \|\phi'(x)b\|_{Y^*}$ where $b = \sum b_j$ and $\|b\|_{Y^*} = 1$.

THEOREM 3.9: *Suppose X is a Köthe function space with X' norming. If X is r-convex, $r > 1$, then for any p, $1 < p < \infty$, we have*

$$D(p,r) \subset D(p,X).$$

Proof: We may suppose that X is r-convex with constant one and so X' is $r' = s$—concave with constant one. We show $Y = l^r$ works in Proposition 3.3. In fact suppose $a_1, \ldots, a_n$ are as in the statement. Then if $\mu_j \geq 0$ the function $F(\mu_1, \ldots, \mu_n) = \|\sum \mu_j^{1/s} a_j\|_{X^*}^s$ is concave on the positive cone of $\mathbb{R}^n$. It follows easily that there exist scalars $c_1, \ldots, c_n$ so that $\sum c_j, \mu_j \geq F(\mu)$ for all $\mu \geq 0$ and $\sum c_j = F(1, \ldots, 1) = 1$. Now pick b_j to be disjoints elements in l^s with norms $c_j^{1/s}$.

The following theorem generalizes the "if" part of (i).

THEOREM 3.10: *Let X be a Köthe function space with X' norming. If X is r-convex for some $r > 1$ and not s-convex for any $s > r$, then*

$$D(p, X) = D(p, r).$$

Proof: $D(p,r) \subset D(p,X)$ is proved in the last Theorem.

The proof of $D(p,X) \subset D(p,r)$ follows the same lines of the proof of part (i) of Theorem 3.3 and we shall omit the details. We only point out that an appropiate generalization of Proposition 3.2 is required:

PROPOSITION 3.11: *Let X be a Köthe function space with X' norming such that X is r-convex for some r, $1 < r < \infty$ and X is not s-convex for any s, $r < s < \infty$. Then, given $\varepsilon > 0$ and a positive integer m, there exists a sequence $\{e_i\}_{i=1}^m$ of pairwise disjoint elements of X such that*

$$(3.12) \qquad (1-\varepsilon)\sum_{i=1}^m a_i^r \le \left\| \sum_{i=1}^m a_i e_i \right\|_X^r \le c_r^r \sum_{i=1}^m a_i^r$$

holds for any sequence $\{a_i\}_{i=1}^m$ of non-negative scalars and c_r the constant of r-convexity of X.

Proof: We define

$$X_{(r)} = \{f\colon f \text{ is measurable and } |f|^{1/r} = g \text{ for some } g \in X\}.$$

Under our hypotheses, it is easy to check that

$$\|f\|_{X_{(r)}} = \inf\left\{ \sum_{i=1}^m \| \, |f_i|^{1/r} \|_X^r \colon |f| = \sum_{i=1}^m |f_i|, f_i \in X_{(r)}, m \ge 1 \right\}$$

is a norm. The space $X_{(r)}$ becomes a Köthe function space when equipped with this norm. Moreover, if c_r is the constant of r-convexity, then

$$(3.13) \qquad c_r^{-r} \| \, |f|^{1/r} \|_X^r \le \|f\|_{X_{(r)}} \le \| \, |f|^{1/r} \|_X^r.$$

For more details see [L,T], p. 54. The space $X_{(r)}$ has the important property of not being q-convex for any $q > 1$. Then, by Proposition 3.2, given $\varepsilon > 0$ and $m > 0$, there is a sequence $\{f_i\}_{i=1}^m$ of positive and disjoint elements of $X_{(r)}$ such that

$$(1-\varepsilon)\sum_{i=1}^m b_i \le \| \sum_{i=1}^m b_i f_i \|_{X_{(r)}} \le \sum_{i=1}^m b_i$$

holds for any sequence $\{b_i\}_{i=1}^m$ of non-negative scalars. Then, since $\{f_i\}$ is pairwise disjoint, we have

$$\left(\sum b_i f_i\right)^{1/r} = \sum b_i^{1/r} f_i^{1/r}. \tag{3.14}$$

Let $e_i = f_i^{1/r}$. Then $e_i \in X$ and, by (3.13) and (3.14), we obtain

$$c_r^{-r} \left\| \sum b_i^{1/r} e_i \right\|_X^r \leq \left\| \sum b_i f_i \right\|_{X_{(r)}} \leq \left\| \sum b_i^{1/r} e_i \right\|_X^r.$$

Thus,

$$(1-\varepsilon) \sum b_i \leq \left\| \sum b_i^{1/r} e_i \right\|_X^r \leq c_r^r \sum b_i.$$

Taking $a_i = b_i^{1/r}$ we get (3.12). ∎

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